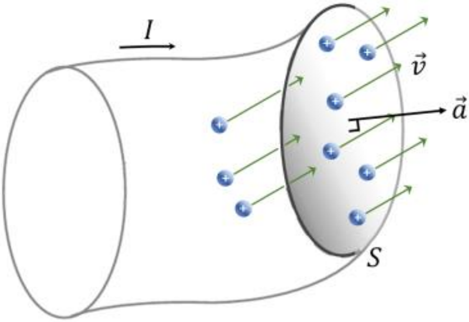
Chapter 4

Electric Currents

4-1 Steady Current

A. CURRENT DENSITY

(1) The amount of charges passing a given point per unit time.



$$I = \frac{dQ}{dt} = \frac{\rho \vec{v} dt \cdot d\vec{a}}{dt} = \rho \vec{v} \cdot \vec{a}$$

where ρ is the volume charge density.

Define the current density (current per unit area)

$$\vec{J} = \frac{dI}{da}\hat{n} = \rho \vec{v}$$

Thus, the current I flowing through any surface $\mathcal{S}$ is just the surface integral

$$I = \int_{\mathcal{S}} \vec{J} \cdot d\vec{a}$$

(2) If we choose the surface to be closed, Gauss's divergence theorem permits us to express I as an integral over the enclosed volume $\mathcal{V}$:

$$I = \oint_{\mathcal{S}} \vec{J} \cdot d\vec{a} = \int_{\mathcal{V}} \nabla \cdot \vec{J}\, d\tau$$

Since the charge is leaving the enclosed volume,

$$I = \frac{dQ}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau$$

so we have

$$\int_{\mathcal{V}} \nabla \cdot \vec{J}\, d\tau = -\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = -\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau$$

This equation gives rise to

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \cdots\cdots \text{continuity equation}$$

This equation is also the precise mathematical statement of local charge conservation.

B. STEADY CURRENT

(1) Electric charge in motion is electric current. Because charge is never created or destroyed, the charge density ρ and the current density $\vec{J}$ always satisfy the continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

If ρ is constant in time, we have

$$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{J} = 0$$

According to Gauss's law,

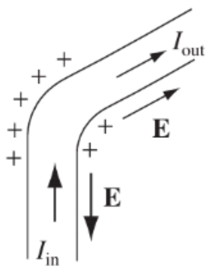
$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \epsilon_0 \nabla \cdot \vec{E}$$

we have

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t}\left(\epsilon_0 \nabla \cdot \vec{E}\right) = -\epsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t} = 0$$

Thus, the electric field $\vec{E}$ is constant in time. The current driven by this electric field is called the steady current.

(2) Suppose that the current into the bend is greater than the current out.



Then charge piles up at the "knee", and this produces a field aiming away from the kink. This field opposes the current flowing in (slowing it down) and promotes the current flowing out (speeding it up) until these currents are equal, at which point there is no further accumulation of charge, and equilibrium is established. *The current is the same all around the circuit.*

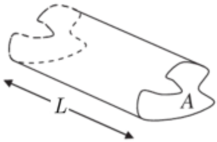
When the current density vector $\vec{J}$ remains constant in time everywhere, no charge piling up occurs, i.e.,

$$\frac{d}{dt}\rho = 0 \Rightarrow \nabla \cdot \vec{J} = 0$$

The current is steady in a wire, i.e., the magnitude of I is the same all along the line.

EXAMPLES:

1. A wire has arbitrary shape of the cross-section A and length L. If we stipulate that the potential is constant over each end, and the potential difference between the ends is φ. Show that the field inside the wire is uniform.



ANSWER:

Within the wire, φ obeys Laplace's equation. The Neumann boundary condition on the side walls are

$\varphi(0) = 0$ and $\varphi(L) = \varphi_0$

$$\frac{\partial\varphi}{\partial n} = 0$$

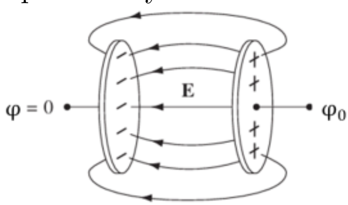
Thus, the potential is

$$\varphi(z) = \frac{\varphi_0 z}{L}$$

The field is

$$\vec{E} = -\nabla\varphi = -\frac{\varphi_0}{L}\hat{z}$$

which is equivalent to a parallel-plate capacitor whose plates are separated by a distance L.



4-2 Ohm's Law and Resistance

A. OHM'S LAW

(1) In most substances, and over a wide range of electric field strengths, we find a *empirical* relation that the current density is proportional to the strength of the electric field that causes it,

$$\vec{J} = \sigma \vec{E} \cdots\cdots \text{Ohm's law}$$

where σ is called the electric conductivity.

(2) Ohm's law describes the motion of charged particles that are accelerated by an electric field but suffer energy and momentum degrading collisions (scattering events) with other particles in a metal. Supposed that charged particles moving through a metal are subject to an external electric field $\vec{E}$ and collisions between nuclei. Thus, the total momentum $\vec{p}$ of a bunch of charge carriers in some volume of a metal is governed by the equation:

$$\frac{d\vec{p}}{dt} = q\vec{E} + \vec{F}_{\text{coll}}$$

Drude assumes that electrons are like a classical ideal gas following the Maxwell-Boltzmann distribution. Thus, when a collision happens, the electron stops dead:

$$\vec{F}_{\text{coll}} = \frac{\Delta\vec{p}}{\tau} = \frac{0 - m\vec{v}_d}{\tau} = -\frac{m\vec{v}_d}{\tau}$$

where τ is the time between collisions and called the relaxation time. $\vec{v}_d$ is called the drift velocity.
Thus, we obtain

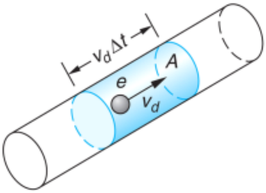
$$\frac{d\vec{p}}{dt} = q\vec{E} - \frac{m\vec{v}_d}{\tau}$$

For a steady current, i.e., $d\vec{p}/dt = 0$, we obtain

$$0 = q\vec{E} - \frac{m\vec{v}_d}{\tau} \Rightarrow \vec{v}_d = \frac{q\tau}{m}\vec{E} = \mu\vec{E}$$

where $\mu = q\tau/m$ is called the mobility.

(3) The current density



$$\vec{j} = \frac{I}{A} = \frac{\Delta Q}{A\Delta t} = \frac{1}{A}\frac{\rho AL}{L/\vec{v}_d} = \rho\vec{v}_d$$

where $L = \vec{v}_d\Delta t$.

According to Drude's theory and Ohm's law, we have

$$\vec{j} = \rho\mu\vec{E} = \sigma\vec{E} \Rightarrow \sigma = \rho\mu = \rho\frac{q\tau}{m} = \frac{\rho q\tau}{m} \text{ ····· electrical conductivity}$$

EXAMPLES:

1. The electron density in copper is $n = 8.5 \times 10^{28}\text{m}^{-3}$. The relaxation time is $\tau \sim 2.5 \times 10^{-14}\text{s}$. Find the electric conductivity of copper.

   ANSWER:

   Consider the number density n of electrons,

   $\rho = -ne$

   then we have

   $$\sigma = \frac{(-ne)(-e)\tau}{m}$$
   $$= \frac{ne^2\tau}{m}$$
   $$= \frac{8.5 \times 10^{28}\text{m}^{-3} \times \left(1.602 \times 10^{-19}\text{C}\right)^2 \times 2.5 \times 10^{-14}\text{s}}{9.109 \times^{-31} \text{kg}}$$
   $$= 6 \times 10^7 (\Omega\text{m})^{-1}$$

2. The density of copper is $8.935\,\text{g/cm}^3$. The electron mobility in copper at room temperature is $\mu = 4.4 \times 10^{-3}\,\text{m}^2/\text{Vs}$. Find the number of free electrons per copper atom.

   ANSWER:

   The expression for the electrical conductivity of a metal, according to Drude model is as shown below:

   $\sigma = \rho\mu = ne\mu$

   $$\Rightarrow n = \frac{\sigma}{e\mu} = \frac{6 \times 10^7 (\Omega\text{m})^{-1}}{1.602 \times 10^{-19}\text{C} \times 4.4 \times 10^{-3}\,\text{m}^2/\text{Vs}} = 8.5 \times 10^{28}\text{m}^{-3}$$

   The number of copper atom in a cubic meter is

$$N_{\text{Cu}} = \frac{8.935 \times \frac{10^6}{10^3}\,\text{kg/m}^3 \times 10^3\,\text{g/kg}}{63.55\ (\text{atomic weight})} \times N_A$$

$$= \frac{8935000}{63.55} \times 6.023 \times 10^{23}$$

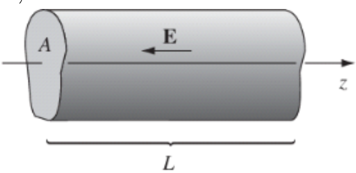
$$= 8.46 \times 10^{28}$$

The number of free electrons in each copper atom is

$$\frac{n}{N_{\text{Cu}}} = \frac{8.5 \times 10^{28}}{8.46 \times 10^{28}} = 1.0$$

B. RESISTANCE

(1) Consider a piece of homogeneous material of conductivity σ and length L, and uniform cross-section A.



Within the conducting material,

$$\vec{J} = \sigma \vec{E}$$

Since the field inside the wire is uniform, the potential difference V between two ends is

$$V = EL$$

The total current is

$$I = \int \vec{J} \cdot d\vec{a} = JA$$

Thus, we obtain

$$J = \frac{I}{A} = \sigma \frac{V}{L} \Rightarrow V = \left(\frac{L}{\sigma A}\right) I \; \cdots\cdots \text{Ohm's law}$$

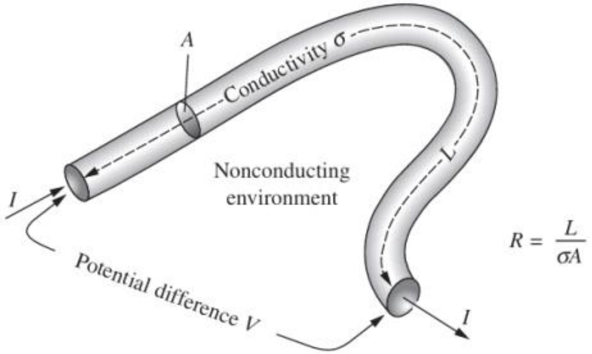
We then define the resistance

$$R = \frac{L}{\sigma A} = \rho_0 \frac{L}{A}$$

where ρ_0 is called the resistivity.

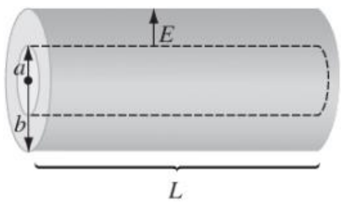
(2) As long as our conductors are surrounded by a nonconducting medium (air, oil, vacuum, etc.), the resistance R between the terminals doesn't depend on the shape, only on the length of the conductor and its cross-

sectional area.



EXAMPLES:

1. Two long coaxial metal cylinders (radii a and b) are separated by material of conductivity σ. If they are maintained at a potential difference V, what current flows from one to the other, in a length L?



ANSWER:

The field between the cylinders is

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$

where λ is the charge per unit length on the inner cylinder. The current is therefore

$$I = \int \vec{J} \cdot d\vec{a} = \sigma \int \vec{E} \cdot d\vec{a} = \sigma \frac{\lambda}{2\pi\epsilon_0 r} \cdot 2\pi r L = \frac{\sigma}{\epsilon_0}\lambda L$$

Meanwhile, the potential difference between the cylinders is

$$V = -\int_a^b \vec{E} \cdot d\vec{s} = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a} \Rightarrow \lambda = \frac{2\pi\epsilon_0}{\ln b/a}V$$

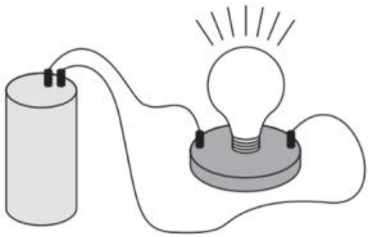
so we have

$$I = \frac{\sigma}{\epsilon_0}\frac{2\pi\epsilon_0}{\ln b/a}VL = \frac{2\pi\sigma L}{\ln b/a}V \Rightarrow R = \frac{2\pi\sigma L}{\ln b/a}$$

4-3 RC Circuit

A. ELECTROMOTIVE FORCE

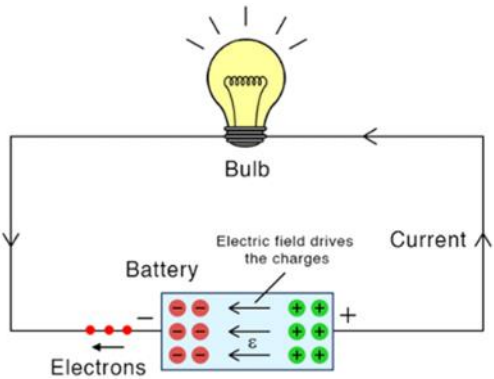
(1) Circuit

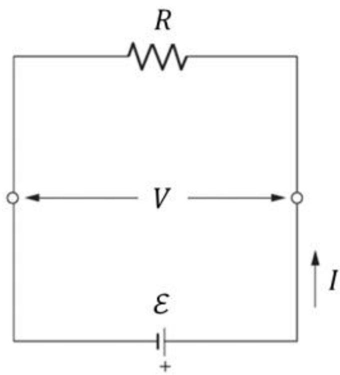


The flow of current in a resistor involves the dissipation of energy. Suppose a steady current I, in amperes, flows through a resistor of R. The rate at which work is done (that is, the power) is therefore

$$P = I^2 R$$

(2) Naturally the steady flow of current in a dc circuit requires some source of energy capable of maintaining the electric field that drives the charge carriers. A battery (source of energy) generates an electric potential difference or a voltage (Voltage V and potential difference are the same. Voltage is a word that is often used in practice).





For historical reasons, we say that these sources of electrical energy (battery) impress an electromotive force (EMF) on the system.

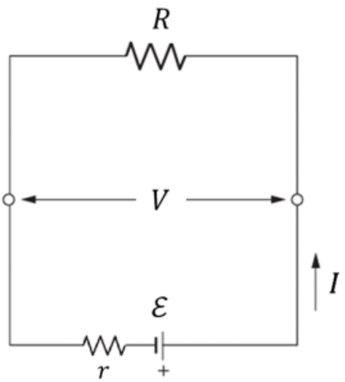
OS:

The term electromotive force was coined by Italian physicist and chemist Alessandro Volta, who invented the electric battery in

1800.

EMF is the energy supplied to the charge.

(3) EMF maintains the potential difference or voltage between two electrodes. When no current is drawn from the battery, the voltage $\mathcal{E}$ between the electrodes of the source is called its electromotive force.

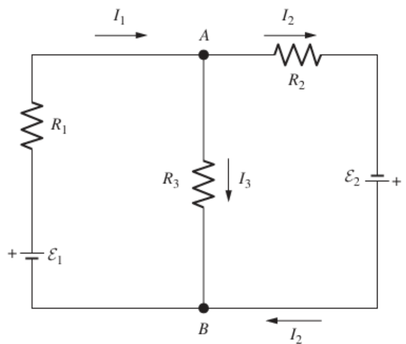


Since in practice the potential difference V will drop only a little below the value $\mathcal{E}$, r plays the role of an effective "internal resistance" which accounts for loss mechanisms within the source itself, i.e.,

$$\mathcal{E} - Ir = V$$

EXAMPLES:

1. A circuit contains two batteries without internal resistance with electromotive force $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. What are the currents in this network?



ANSWER:

$I_1 - I_2 - I_3 = 0$
$\mathcal{E}_1 - R_1 I_1 - R_3 I_3 = 0$
$\mathcal{E}_2 + R_3 I_3 - R_2 I_2 = 0$

Thus, we obtain

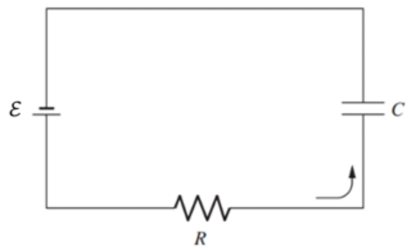
$$I_1 = \frac{\mathcal{E}_1 R_2 + \mathcal{E}_1 R_3 + \mathcal{E}_2 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}$$

$$I_2 = \frac{\mathcal{E}_2 R_1 + \mathcal{E}_2 R_3 + \mathcal{E}_1 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}$$

$$I_3 = \frac{\mathcal{E}_1 R_2 - \mathcal{E}_2 R_1}{R_1 R_2 + R_2 R_3 + R_1 R_3}$$

B. RC CIRCUIT

(1) An RC circuit is a circuit containing resistance and capacitance.



The Kirchhoff loop equation for the series RC circuit and the dc current is

$$\mathcal{E} = \frac{q}{C} + RI$$

Since

$$I = \frac{dq}{dt}$$

we have

$$\frac{q}{C} + R\frac{dq}{dt} = \mathcal{E} \Rightarrow \frac{dq}{dt} = \frac{C\mathcal{E}}{RC} - \frac{q}{RC}$$

This differential equation can be integrated to find an equation for the charge on the capacitor as a function of time.

$$\frac{dq}{C\mathcal{E} - q} = \frac{dt}{RC}$$

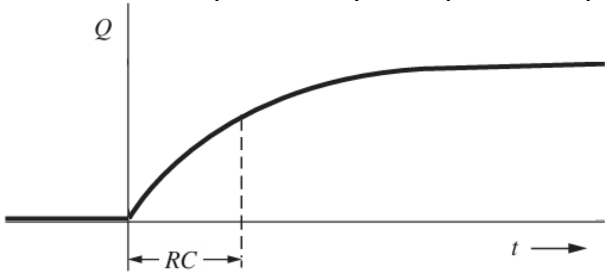
$$\int_0^Q \frac{dq}{C\mathcal{E} - q} = \frac{1}{RC}\int_0^t dt$$

$$-\ln(C\mathcal{E} - q)\Big|_0^Q = -\ln\frac{C\mathcal{E} - Q}{C\mathcal{E}} = \frac{t}{RC}$$

$$\frac{C\mathcal{E} - Q}{C\mathcal{E}} = e^{-t/RC}$$

Let $Q_0 = C\mathcal{E}$. We have

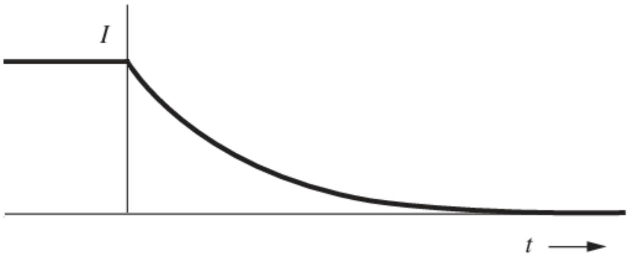
$$Q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right) = Q_0\left(1 - e^{-t/RC}\right)$$



The quantity RC that appears in the exponent is called the time constant of the circuit.

(2) The current through the resistor can be found by taking the time derivative of the charge.

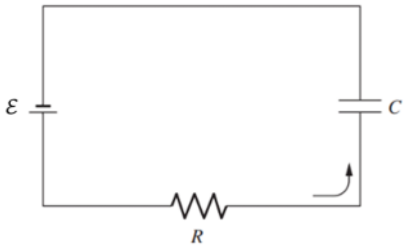
$$I = \frac{dQ}{dt} = \frac{Q_0}{RC}e^{-t/RC} = \frac{\mathcal{E}}{R}e^{-t/RC} = I_0 e^{-t/RC}$$



4-4 Electrostatic Energy

A. ENERGY STORED IN THE ELECTRIC FIELD

(1) In a simple RC circuit, the power supplied by the battery is



$$I\mathcal{E} = I^2R + I\frac{Q}{C}$$

The first term is the power dissipated as heat. The second term is the rate of increase of the electric energy.

Thus, if U is the stored electric energy,

$$\frac{dU}{dt} = I\frac{Q}{C} = \frac{1}{C}Q\frac{dQ}{dt} = Q\frac{dV}{dt} \Rightarrow dU = \frac{1}{C}QdQ = CVdV$$

Clearly, $U = 0$ when $Q = 0$, then

$$U = \int dU = \frac{1}{C}\int Q\,dQ = \frac{1}{2C}Q^2 = \frac{1}{2}CV^2$$

It takes work to charge up a capacitor

$$W = U = \frac{1}{2}CV^2$$

(2) The electric energy W expressed in terms of V and ρ.

$$\frac{dW}{dt} = I\frac{Q}{C} = IV = V\frac{dQ}{dt}$$

Since

$$Q = \int_{\mathcal{V}} \rho\,d\tau$$

we have

$$V\frac{dQ}{dt} = \frac{d}{dt}\int_{\mathcal{V}} V\rho\,d\tau = \int_{\mathcal{V}} V\frac{\partial\rho}{\partial t}d\tau$$

Since

$$\frac{d}{dt}\int_{\mathcal{V}}\rho V\,d\tau = \int_{\mathcal{V}}\rho\frac{\partial V}{\partial t}d\tau + \underbrace{\int_{\mathcal{V}}\frac{\partial\rho}{\partial t}V\,d\tau}_{=V\frac{dQ}{dt}}$$

Similarly, we can obtain

$$Q\frac{dV}{dt} = \frac{d}{dt}\int_{\mathcal{V}}V\rho\,d\tau = \int_{\mathcal{V}}\rho\frac{\partial V}{\partial t}d\tau$$

Thus, we get

$$\frac{d}{dt}\int_{\mathcal{V}}\rho V\,d\tau = \underbrace{\int_{\mathcal{V}}\rho\frac{\partial V}{\partial t}d\tau}_{=Q\frac{dV}{dt}} + \underbrace{\int_{\mathcal{V}}\frac{\partial\rho}{\partial t}V\,d\tau}_{=V\frac{dQ}{dt}}$$

Since

$$\frac{1}{C}Q\frac{dQ}{dt} = Q\frac{dV}{dt} \Rightarrow V\frac{dQ}{dt} = Q\frac{dV}{dt}$$

we have

$$\frac{d}{dt}\int_{\mathcal{V}}\rho V\,d\tau = 2\int_{\mathcal{V}}\rho\frac{\partial V}{\partial t}d\tau \Rightarrow \int_{\mathcal{V}}\rho\frac{\partial V}{\partial t}d\tau = \frac{1}{2}\frac{d}{dt}\int_{\mathcal{V}}\rho V\,d\tau$$

Thus, we obtain

$$\frac{dW}{dt} = \frac{1}{2}\frac{d}{dt}\int_{\mathcal{V}}\rho V\,d\tau \Rightarrow W = \frac{1}{2}\int_{\mathcal{V}}\rho V\,d\tau$$

(3) The energy expressed in terms of $\vec{E}$.

Using Gauss's law, we have

$$W = \frac{1}{2}\int_{\mathcal{V}}\rho V\,d\tau = \frac{1}{2}\int_{\mathcal{V}}\epsilon_0\left(\nabla\cdot\vec{E}\right)V\,d\tau$$

Since

$$\nabla\cdot\left(\vec{E}V\right) = \left(\nabla\cdot\vec{E}\right)V + \vec{E}\cdot\nabla V$$

thus, we have

$$\begin{aligned}W &= \frac{\epsilon_0}{2}\left[\int_{\mathcal{V}}\nabla\cdot\left(\vec{E}V\right)d\tau - \int_{\mathcal{V}}\vec{E}\cdot\nabla V\,d\tau\right]\\ &= \frac{\epsilon_0}{2}\left[\oint_{\mathcal{S}}\vec{E}V\cdot d\vec{a} - \int_{\mathcal{V}}\vec{E}\cdot\nabla V\,d\tau\right]\\ &= \frac{\epsilon_0}{2}\left[\oint_{\mathcal{S}}\vec{E}V\cdot d\vec{a} + \int_{\mathcal{V}}\vec{E}\cdot\vec{E}\,d\tau\right]\\ &= \frac{\epsilon_0}{2}\left[\oint_{\mathcal{S}}\vec{E}V\cdot d\vec{a} + \int_{\mathcal{V}}E^2\,d\tau\right]\end{aligned}$$

Since at large distances from the charge, $\vec{E}$ goes like $1/r^2$ and V like

$1/r$, while the surface area grows like r^2; then, the surface integral goes down like $1/r$. Thus, we obtain

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau = \frac{1}{2}CV^2$$

EXAMPLES:

1. Find the energy of a uniformly charged spherical shell of total charge q and radius R.

   ANSWER:

- Method I:

$$W = \frac{1}{2}\int \sigma V\, da = \frac{1}{2}\int \frac{q\sigma}{4\pi\epsilon_0 R}\, da = \frac{1}{2}\frac{q}{4\pi\epsilon_0 R}q = \frac{q^2}{8\pi\epsilon_0 R}$$

- Method II:

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau$$

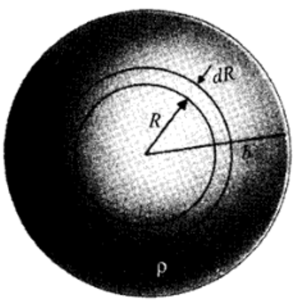
$$= \frac{\epsilon_0}{2}\int \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 r^2 \sin\theta\, dr d\theta d\phi$$

$$= \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int_R^\infty \frac{1}{r^2} dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{1}{R}\cdot 2\cdot 2\pi$$

$$= \frac{q^2}{8\pi\epsilon_0 R}$$

2. Find the energy required to assemble a uniform sphere of charge of radius b and volume charge density ρ.



   ANSWER:

- Method I:

$$V = \frac{q}{4\pi\epsilon_0 R}$$

$$dW = Vdq = \frac{q}{4\pi\epsilon_0 R}\rho 4\pi R^2 dR = \frac{q\rho}{\epsilon_0} RdR$$

Since

$$q = \frac{4}{3}\pi R^3 \rho$$

we have

$$dW = \frac{\rho}{\epsilon_0}\frac{4}{3}\pi R^3 \rho R dR = \frac{4\pi}{3\epsilon_0}\rho^2 R^4 dR$$

The total work required to assemble a uniform sphere of charge is

$$W = \int dW = \int_0^b \frac{4\pi}{3\epsilon_0}\rho^2 R^4\, dR = \frac{4\pi}{15\epsilon_0}\rho^2 b^5$$

- Method II:

$$W = \frac{1}{2}\int \rho V\, d\tau = \frac{\rho}{2}\int_0^b V 4\pi R^2\, dR$$

Since

$$\vec{E} = \begin{cases} \dfrac{q}{4\pi\epsilon_0 R}\hat{r} = \dfrac{\rho b^3}{3\epsilon_0 R^2}\hat{r} \quad , & R \geq b \\ \dfrac{q}{4\pi\epsilon_0 R^2}\hat{r} = \dfrac{\rho R}{3\epsilon_0}\hat{r} \quad , & 0 < R \leq b \end{cases}$$

Consequently, we obtain

$$\begin{aligned} V &= -\int_\infty^R \vec{E}\cdot d\vec{s} \\ &= -\int_\infty^b \vec{E}\cdot d\vec{s} - \int_b^R \vec{E}\cdot d\vec{s} \\ &= -\int_\infty^b \frac{\rho b^3}{3\epsilon_0 R^2} dR - \int_b^R \frac{\rho R}{3\epsilon_0} dr \\ &= \frac{\rho}{3\epsilon_0}\left(\frac{b^3}{b} - \frac{b^3}{\infty} - \frac{R^2}{2} + \frac{b^2}{2}\right) \\ &= \frac{\rho}{6\epsilon_0}\left(3b^2 - R^2\right) \end{aligned}$$

Thus, we get

$$
\begin{aligned}
W &= \frac{\rho}{2}\int_0^b \frac{\rho}{6\epsilon_0}\left(3b^2 - R^2\right)4\pi R^2\, dR \\
&= \frac{\pi\rho^2}{3\epsilon_0}\int_0^b \left(3b^2R^2 - R^4\right) dR \\
&= \frac{\pi\rho^2}{3\epsilon_0}\left(b^5 - \frac{b^5}{5}\right) \\
&= \frac{4\pi}{15\epsilon_0}\rho^2 b^5
\end{aligned}
$$